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Independent Sets of Knots and Singularity of Interpolation Matrices

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1. INTRODUCTION

One of the purposes of this paper is to give a new proof of a theorem on singularity of Birkhoff interpolation matrices (Lorentz [3, 4], Karlin and Karon [2])—our Theorem 22 (compare Section 9 for the history of this theorem). A new proof is necessary, because that of [3] gives only ordinary singularity and is somewhat sketchy, and paper [2], which purports to prove strong singularity, contains a serious error. Our proof is almost identical with one given in the 1975 report [5]. The novel feature of our paper is the switch from the system of powers $S = \{1, x, ..., x^n\}$ to an "arbitrary" systems of functions $S = \{g_0, ..., g_n\}$. This is achieved by considering "Birkhoff systems." The importance of this notion, first of all, is that the Atkinson–Sharma theorem of regularity is valid exactly for Birkhoff systems. On the other hand, all singularity theorems known at present (see [6, Sect. 5]) are valid also for Birkhoff systems. A Birkhoff system is always an extended Chebyshev system. The converse, although not true, holds locally (see Section 6). This allows the conclusion that all known singularity theorems apply to extended Chebyshev systems. This indirect approach seems to be necessary, since the earlier proofs [3, 2] are only for algebraic polynomials.

There are two ways to prove the basic singularity theorem. The method of independent knots has been introduced for this purpose by Lorentz and Zeller [7] and further developed by Lorentz [3, 5, 11].

The other approach is by means of coalescence, a notion used implicitly by Ferguson [1], developed by Karlin and Karon [2] and Lorentz and Zeller [8]. The disadvantage of the method of coalescence in the proof of our theorem has been pointed out by S. D. Riemenschneider. This method requires differentiability of functions $g_k \in S$ of very high order. If α is the coefficient of collision of two rows (see [6, Sect. 3]), then the application of

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the method requires differentiability of g_k of order at least α. (Derivatives of orders $>n$ disappear from the formulas only *after* coalescence). Since α could be as large as $\frac{1}{4}n^2$, the method requires that all $g_k \in S$ be differentiable of this order! This is one of the reasons why we use the method of independent knots in this paper. The other is that it is of some interest in itself.

The basic notions of Birkhoff interpolation (see, for example, [6]) are as follows.

Let $S = \{g_0, ..., g_n\}$ be a set of linearly independent functions on $[a, b]$, that are n times continuously differentiable. Let $E = (e_{ik})$ be an $m \times (n+1)$ interpolation matrix of 0's and 1's which is normal (that is, has $n+1$ ones), let $X: x_1 < \cdots < x_m$ be a set of knots in $[a, b]$. Linear combinations $P = \sum_0^n a_k g_k$ we call polynomials. A Birkhoff problem is to find a polynomial P satisfying $P^{(k)}(x_i) = c_{ik}$, where c_{ik} are given numbers, and this relation is required for all pairs (i, k) for which $e_{ik} = 1$. If this problem is solvable for all choices of c_{ik}, the pair E, X is *regular* (*poised*), otherwise it is *singular*. The former is the case if and only if each P annihilated by E, X (that is, satisfying $P^{(k)}(x_i) = 0$ for $e_{ik} = 1$) is identically zero, and if and only if the $(n+1) \times (n+1)$ determinant

$$D(E, X) = \{ g_0^{(k)}(x_i), ..., g_n^{(k)}(x_i); e_{ik} = 1 \} \tag{1.1}$$

is not zero. Matrix E is *regular* if all possible pairs E, X are regular. Consequently, E is *singular* if for some X, the determinant (1.1) vanishes. Further, E is *strongly singular* if $D(E, X)$ changes sign for sets of knots X in $[a, b]$, and *conditionally regular*, if $D(E, X) \neq 0$ for some X in $[a, b]$.

Let M_k be the number of ones in columns $0, 1, ..., k$ of E. Matrix E satisfies the Pólya (Birkhoff) condition if $M_k \geqslant k+1$, $k = 0, ..., n$ (resp. $M_k \geqslant k+2$, $k = 0, ..., n-1$). A Pólya (Birkhoff) matrix E is a normal matrix that satisfies the Pólya (Birkhoff) condition. The following two theorems are known if $S = \{1, x, ..., x^n\}$:

THEOREM A (Birkhoff–Ferguson–Nemeth). *A normal interpolation matrix E is conditionally regular if and only if E is a Pólya matrix.* (The "only if" statement holds for arbitrary S.)

A *decomposition* of E is a vertical decomposition into normal interpolation matrices. The *canonical decomposition* of E is the finest decomposition of this kind. A *sequence* Σ of E is a maximal block of 1's in one of the rows of E. If $e_{i_0 q} = 1$ is the first one of Σ, we call Σ *supported*, if there exist $e_{ik} = 1$ with $k < q$ and both $i < i_0$ and $i > i_0$. A subclass are *essentially supported* sequences Σ of E, which are supported within the matrix of the canonical decomposition of E to which Σ belongs.

THEOREM B (Atkinson–Sharma). *A Pólya matrix without odd essentially supported sequences is regular with respect to the set* $S = \{1, x, ..., x^n\}$.

The proof of this is based on the following form of Rolle's theorem:

THEOREM C (Rolle's Theorem). *Let* f *be a* p *times continuously differentiable function on* $[a, b]$, $p \geqslant 1$, *then between any two adjacent zeros of* f *there is an odd number of zeros of* f' *(counting multiplicities), or a zero of* $f^{(p)}$.

2. ROLLE EXTENSIONS

Rolle's theorem can be useful also in situations when odd supported sequences are present, but there are only few of them or when their influence is weak. To pursue this idea we need the notion of Rolle extensions.

Let E be an $m \times (n+1)$ interpolation matrix, and f be an n-times differentiable function on $[a, b]$ which is annihilated by E and $X = (x_1, ..., x_m) \subset [a, b]$, that is, let f satisfy

$$f^{(l)}(x_i) = 0, \qquad \text{whenever } e_{il} = 1 \text{ in } E. \tag{2.1}$$

The pair (E, X) defines Eqs. (2.1) and conversely. We shall often identify this pair with the equations. From the zeros of f and its derivatives specified by (2.1), we can derive further zeros by means of Rolle's theorem. A selection of a complete set of such zeros is called a *Rolle extension of Eqs.* (2.1). This is a pair $(\tilde{E}, \tilde{X})$ with the corresponding equations

$$f^{(l)}(\tilde{x}_i) = 0, \qquad \tilde{e}_{il} = 1 \qquad \text{in } \tilde{E}, \tag{2.2}$$

which contain all of (2.1), but in general also some additional equations. The extension is not unique. The formal definition is as follows.

A *Rolle extension* $\mathscr{R} = (\tilde{E}, \tilde{X})$ for a function f annihilated by the pair (E, X) (or for Eqs. (2.1)) is obtained by selecting by induction *Rolle extensions* $\mathscr{R}_k = (E^k, X^k)$ for each $k = 0, 1, ..., n$. Here, E^k is an $m_k \times (n - k + 1)$ matrix (with columns numbered k, $k+1, ..., n$), and equations of $\mathscr{R}_k$ contain all Eqs. (2.1) with $l \geqslant k$.

Equations of $\mathscr{R}_0$ are simply the set (2.1). If $\mathscr{R}_0, ..., \mathscr{R}_k$ have been already selected, we choose a pair $(E^{k+1}, X^{k+1}) = \mathscr{R}_{k+1}$ according to the following prescriptions:

1. It contains all equations of (E^k, X^k) for derivatives $f^{(l)}$, $l \geqslant k+1$.

2. Between any two *adjacent* zeros $\alpha < \beta$ of $f^{(k)}$ belonging to $\mathscr{R}_k$, we select, *if possible*, a zero of $f^{(k+1)}$, not listed in $\mathscr{R}_k$. This could be (a) a new zero ξ of $f^{(k+1)}$, or (b) a new zero ξ of $f^{(l)}$, $l > k+1$, if equations $f^{(k+1)}(\xi) = 0, ..., f^{(l-1)}(\xi) = 0$ are contained in $\mathscr{R}_k$. In this case the

multiplicity of ξ as zero of $f^{(k+1)}$, which is acknowledged by $\mathscr{R}_k$, is increased by at least one.

3. If for a pair $\alpha < \beta$ this is *impossible*, we register a *loss* and do not add a new equation to $\mathscr{R}_k$ for the pair α, β.

The Rolle extension $\mathscr{R}$ consists of all equations contained in all $\mathscr{R}_k$, $k = 0, 1,..., n$. In other words, Eqs. (2.2) of $\mathscr{R}$ for a given l, $l = 0,..., n$ consist of the equations for $f^{(l)}$ which belong to the extension $\mathscr{R}_l$. A Rolle extension constructed without losses at any of its steps is called *maximal.* An extension $\mathscr{R}$ in which 2(b) has never been used, is called an extension *without duplication.* A function f may have many Rolle extensions $\mathscr{R}$, some of them may be maximal, while others are not maximal.

Some properties of Rolle extensions are immediate consequences of the selection procedure. A root η of $f^{(k)}$ in $\mathscr{R}_k$ of multiplicity σ, is also a root of $f^{(k+1)}$ in $\mathscr{R}_{k+1}$, of multiplicity exactly $\sigma - 1$. A new root ξ of $f^{(k+1)}$ selected by 2(a) or 2(b), has in $\mathscr{R}_{k+1}$ a multiplicity not less than $\tau + 1$, if τ is the multiplicity of $f^{(k+1)}(\xi) = 0$, acknowledged by $\mathscr{R}_k$ ($\tau = 0$ in case 2(a)). This multiplicity will be greater than $\tau + 1$ exactly when $\mathscr{R}_k$ contains also the equation $f^{(l+1)}(\xi) = 0$ (see 2(b)).

It follows also that the matrix E^{k+1} contains as a submatrix the last $n - k$ columns of E^k (hence also the last $n - k$ columns of E). In part (2) of the construction, for given $\alpha < \beta$, ξ can be found (there is no loss) if we assume that rows of E for which $\alpha < x_i < \beta$, have no odd supported sequences. This follows at once from Rolle's Theorem C. In particular:

LEMMA 1. *If the matrix E has no odd supported sequences, then all Rolle extensions of a function f, annihilated by E, X, are maximal.*

We also have:

LEMMA 2. *A maximal Rolle extension $\mathscr{R}$ of a pair E, X, has the properties*: (i) *If E satisfies the Pólya condition $M_l \geqslant l + 1$ for $0 \leqslant l \leqslant k_0$, then also all matrices E^k, $k \leqslant k_0$ satisfy this condition for $l \leqslant k$*; (ii) *if E is a Pólya matrix, then all E^k are Pólya matrices.*

Under certain conditions, we can find a simple formula for the number of equations for $f^{(k)}$ in $\mathscr{R}$. Let m_k, M_k, $k = 0, 1,..., n$ be the Pólya functions of E, let $\mu_{-1} = 0$ and

$$\mu_k = (\cdots ((m_0 - 1)_+ + m_1 - 1)_+ \cdots + m_{k-1} - 1)_+ + m_k, \quad k = 0,..., n. \tag{2.3}$$

We have then

$$\mu_0 = m_0, \qquad \mu_k = (\mu_{k-1} - 1)_+ + m_k, \qquad k = 1,..., n. \tag{2.4}$$

In particular, *if E is a Pólya matrix*, we can drop all subscripts in these formulas; then

$$\mu_k = M_k - k, \qquad k = 0,\dots, n. \tag{2.5}$$

LEMMA 3. *Let $\mathscr{R}$ be a maximal Rolle extension of Eqs.* (2.1) *obtained without duplication. Then the number of equations for $f^{(k)}$ in $\mathscr{R}_k$ is exactly μ_k.*

Proof. Let this be true for some k. Then the number of adjacent pairs of zeros $\alpha < \beta$ of $f^{(k)}$ in $\mathscr{R}_k$ is $(\mu_k - 1)_+$, hence the number of different zeros of $f^{(k+1)}$ in the construction of $\mathscr{R}_{K+1}$ is $(\mu_k - 1)_+ + m_{k+1} = \mu_{k+1}$.

COROLLARY 4. *Let f, annihilated by E, X be such that at each step $k = 0,\dots, n$, Rolle zeros of f can be chosen to avoid all x_i, $i \neq i_0$. Let row i_0, $1 < i_0 < m$ of E have no odd supported sequences, and let E satisfy the Birkhoff condition for all k with $0 \leqslant k < k_0$. There is then a Rolle extension $\mathscr{R}$ of E, X having for each $k \leqslant k_0$ either μ_k or $\mu_k - 1$ Rolle zeros. The last case can happen only if $e_{i_0 k} = 1$ belongs to an (even) supported sequence.*

Indeed, in constructing $\mathscr{R}_k$, we always have $\mu_k = (\mu_{k-1} - 1)_+ + m_k$ zeros until there is duplication at some level k. This means that in (2) we have $f^{(k-1)}(\alpha) = f^{(k-1)}(\beta) = 0$, $\alpha < x_{i_0} < \beta$ and that Rolle's theorem produces a zero $f^{(l)}(x_{i_0}) = 0$, $l > k+1$, according to (2b). Then for $k \leqslant j < l$ we have $\mu_j - 1 = (\mu_{j-1} - 2)_+ + m_j$ Rolle zeros, for $j = l$ again μ_l zeros, and so on.

3. AN AUXILLIARY THEOREM

In Section 8 we shall need relations between the number of Rolle zeros of E and of different matrices derived from E. Let $1 < i < m$ be fixed. Let E' be the $i_0 \times (m+1)$ matrix consisting of the rows $i = 1, 2,\dots, i_0$ of E, and E'' be the $(m - i_0 + 1) \times (n+1)$ matrix consisting of the rows $i = i_0,\dots, m$ of E. If m_k, m'_k and m''_k are the respective Pólya functions for E, E' and E'', then

$$m_k = m'_k + m''_k - e_{i_0,k}, \qquad k = 0, 1,\dots, n. \tag{3.1}$$

Let μ_k, μ'_k and μ''_k be the numbers defined by (2.3) for the matrices E, E' and E'', respectively.

THEOREM 5. (i) *Assume that the matrix E satisfies the Birkhoff condition for its columns $k = 0, 1,\dots, k_0 - 1$. Then*

$$\mu_k \geqslant \mu'_k + \mu''_k - e_{i_0,k}, \qquad k = 0,\dots, k_0. \tag{3.2}$$

(ii) *Moreover, if* $e_{i_0,k_0-1}=0$, *then equality holds in* (3.2) *if and only if* $e_{i,k}=0$, $0\leqslant k<k_0$, *for either* (a) *all* $i<i_0$, *or else* (b) *all* $i>i_0$.

Proof. By the Birkhoff condition and (2.5), $\mu_k\geqslant 2$ for $0\leqslant k<k_0$. Thus, $(\mu_k-1)_+=\mu_k-1$, $k=0,\ldots,k_0$. The proof is carried out by induction. It is clear that (3.2) holds for $k=0$. Let

$$\sigma_k=\mu_k-\mu'_k-\mu''_k+e_{i_0,k}. \tag{3.3}$$

By (2.4) and (3.1) we have

$$\sigma_k=\mu_k-1-(\mu'_{k-1}-1)_+-(\mu''_{k-1}-1)_+,\qquad k=0,\ldots,k_0-1. \tag{3.4}$$

LEMMA 6. (i) *One has* $\sigma_k\geqslant 0$, $k=0, k_0,\ldots,k_0$. (ii) *If for some* $k\leqslant k_0$, $\sigma_k=0$, *then* $\sigma_l=0$ *for all* $l\leqslant k$.

Proof. Clearly we have one of the three (not mutually exclusive) cases.

Case 1. $\mu'_{k-1},\mu''_{k-1}\leqslant 1$. Then from (3.4), $\sigma_k=\mu_{k-1}-1\geqslant 1$, and we have (i).

Case 2. One of the μ'_{k-1}, μ''_{k-1} is equal zero, and the other is $\geqslant$1, for example, let $\mu'_{k-1}\geqslant 1$, $\mu''_{k-1}=0$. Then, again by (3.4), $\sigma_k=\mu_{k-1}-\mu'_{k-1}$. Since $e_{i_0,k-1}\leqslant\mu''_{k-1}$, we have $e_{i_0,k-1}=0$, and by (3.3), $\sigma_{k-1}=\mu_{k-1}-\mu'_{k-1}$, hence we have $\sigma_{k-1}=\sigma_k$.

Case 3. Let $\mu'_{k-1},\mu''_{k-1}\geqslant 1$. Then from (3.3), (3.4),

$$\begin{aligned}\sigma_{k-1}&=\mu_{k-1}-\mu'_{k-1}-\mu''_{k-1}+e_{i_0,k-1},\\ \sigma_k&=\mu_{k-1}-\mu'_{k-1}-\mu''_{k-1}+1.\end{aligned} \tag{3.5}$$

In this case, $\sigma_{k-1}\leqslant\sigma_k$.

Now (i) follows by induction, from $\sigma_0\geqslant 1$. After (i) has been established, if $\sigma_k=0$, case 1 cannot happen, and in the other two cases we have $\sigma_{k-1}=0$. Thus $\sigma_l=0$, $l\leqslant k$.

LEMMA 7. *Let* $\sigma_k=0$, $k>0$, *and let the Birkhoff condition be satisfied for* E *for all columns* $l\leqslant k$. (i) *If* $e_{i_0,k-1}=0$, *then either* $\mu'_{k-1}=\mu_{k-1}\geqslant 2$, $\mu''_{k-1}=0$, *or* $\mu'_{k-1}=0$, $\mu''_{k-1}=\mu_{k-1}\geqslant 2$. (ii) *If* $\mu''_k=e_{i_0k}=m''_k$ (=0 *or* 1), *then* $\mu'_{k-1}=\mu_{k-1}\geqslant 2$, $\mu''_{k-1}=e_{i_0,k-1}=m''_{k-1}$ (=0 *or* 1).

Proof. (i) Case 1 of Lemma 6 is impossible, and also Case 3, since Eqs. (3.5) together with $\sigma_{k-1}=\sigma_k=0$ would imply $e_{i_0,k-1}=1$. Hence one of the numbers $\mu'_{k-1}\mu''_{k-1}$ is $\geqslant$2, and the other is zero. Let, for example, $\mu'_{k-1}\geqslant 2$. Then (3.4) gives $0=\sigma_k=\mu_{k-1}-\mu'_{k-1}$, and (i) follows.

(ii) We need to consider only the case $e_{i_0,k-1}=1$. From $e_{i_0k}=(\mu''_{k-1}-1)_+ + m''_k$ we derive $(\mu''_{k-1}-1)_+=0$, that is, $\mu''_{k-1}=0$ or $=1$. Then

$$\mu''_{k-1}=(\mu''_{k-2}-1)_+ + m''_{k-1}$$

and $m''_{k-1}\geqslant 1$ imply $\mu''_{k-1}=1=m''_{k-1}$, which is the second part of assertion (ii). The first part follows from $0=\sigma_k=\mu_k-\mu'_k$.

It is clear that from (i), (ii), and a statement symmetric to (ii), one derives Theorem 5 by induction.

4. Markov's Inequality and Applications

Construction of independent sets of knots in Section 5 will be based on a weak form of Markov's inequality. This inequality makes it possible to guarantee (Theorem 11) the existence of a Rolle zero of a derivative that is not too close to the given zeros of the function.

Let $S=\{g_0,..., g_n\}$ be a system of n times continuously differentiable functions on $[a,b]$. In the rest of this paper we shall always assume that the functions g_k are *linearly independent on each subinterval* $[a_1,b_1]$ *of* $[a,b]$. For example, Birkhoff systems (see Section 6) have this property.

But we need more. For each $k=1,..., n$, let the *reduced set of derivatives* $S^{(k)}$ *for* $[a_1,b_1]$ consist of those $g_j^{(k)}$ that are not identically zero on $[a_1,b_1]$. We shall assume that *the sets* $S^{(k)}$ *with respect to* $[a,b]$ *consist of linearly independent functions on each subinterval of* $[a,b]$. This assumption is less restrictive than it might appear: each set S has this property locally. More exactly:

Proposition 8. *Let the functions of S be linearly independent on each subinterval of* $[a,b]$. *Then there exists a new basis (also denoted by* $g_0,..., g_n$*) in the linear hull of S and an interval* $[a_0,b_0]\subset[a,b]$, *for which all reduced sets* $S^{(k)}$ *are linearly independent on each subinterval of* $[a_0,b_0]$.

Proof. We construct the basis $g_0,..., g_n$ and the interval $[a_0,b_0]$ by induction. Let the required conditions be satisfied for $S^{(1)},..., S^{(k-1)}$ on $I_{k-1}=[a_{k-1},b_{k-1}]$. This means that there is a basis $g_0,..., g_n$ for which $g_j^{(k-1)}\equiv 0$, $j=0,..., p-1$ on I_{k-1}, while $g_p^{(k-1)},..., g_n^{(k-1)}$ are linearly independent on each subinterval $[\alpha,\beta]$ of I_{k-1}. We consider two cases: (1) On no subinterval of I_{k-1}, does the linear hull $\text{lin}\, S^{(k-1)}$ contain constants. Then we take $I_k=[a_k,b_k]=I_{k-1}$. None of the functions $g_j^{(k)}$, $j\geqslant p$ can vanish identically on $[\alpha,\beta]\subset I_k$, hence $S^{(k)}=\{g_p^{(k)},..., g_n^{(k)}\}$ is the reduced set of derivatives for $[\alpha,\beta]$. If there is a relation $a_p g_p^{(k)}+\cdots+a_n g_n^{(k)}\equiv 0$ on $[\alpha,\beta]$, then, integrating, $a+a_p g_p^{(k-1)}+\cdots+a_n g_n^{(k-1)}\equiv 0$. Here $a=0$, since $\text{lin}\, S^{(k-1)}$ does not contain constants, and by inductive assumption $a_j=0$.

$j \geq p$. Thus, $g_j^{(k)}$, $j \geq p$ are linearly independent on $[\alpha, \beta]$. (2) If lin $S^{(k-1)}$ contains constants on some subinterval of I_{k-1}, let I_k be this subinterval. Changing $g_p, \ldots, g_n$ to some other basis, we can assume that $g_p^{(k-1)} \equiv 1$ on I_k, then the linear hull of $g_{p+1}^{(k-1)}, \ldots, g_n^{(k-1)}$ does not contain constants on any subinterval $[\alpha, \beta]$ of I_k, and as before, $g_{p+1}^{(k)}, \ldots, g_n^{(k)}$ are linearly independent on $[\alpha, \beta]$. At the end, $[a_0, b_0] = I_n$, and the g_k are the elements of the last basis.

For sets S with the above properties, we have

THEOREM 9 ("Markov's inequality"). *For each $l > 0$ there is a constant C_l which depends upon l and S and decreases as a function of l, with the property that for each linear combination P of the functions g_k, and each subinterval $[\alpha, \beta]$ of $[a, b]$ of length $\geq l$,*

$$\|P'\| \leq C_l \|P\|_{[\alpha,\beta]}, \qquad \|P\|_{[\alpha,\beta]} = \max_{\alpha \leq x \leq \beta} |P(x)|. \tag{4.1}$$

Proof. We can subdivide $[a, b]$ into intervals $I_j = [a + j\delta, a + (j+1)\delta]$, $j = 1, \ldots, p$ in such a way that each interval $[\alpha, \beta]$ of length $\geq l$ contains one of the I_j (it is sufficient to take $\delta = (b-a)/p \leq \frac{1}{2}l$). The norm $\|P\|_{[\alpha,\beta]}$ of the restriction of P to $[\alpha, \beta]$ is not less than the norm $\|P\|_j$ of P in $C[I_j]$. Since the correspondence $\sum_0^n c_k g_k \to \sum_0^n c_k g_k'$ maps the $(n+1)$-dimensional linear space spanned by the g_k in $C[I_i]$ linearly into the space spanned by the g_k' in $C[I_j]$ linearly into the space spanned by the g_k' in $C[a, b]$, it has a finite norm N_j. Therefore,

$$\|P'\| \leq N_j \|P\|_j \leq \max_j N_j \|P\|_{[\alpha,\beta]} = C_l \|P\|_{[\alpha,\beta]}.$$

The constants C_l will decrease as functions of l, if we chose each of them to be best possible in (4.1).

LEMMA 10. *For each $l > 0$ there is a number $d = d(l)$, $0 < d \leq \frac{1}{2}$, with the property that if $P(\alpha) = P(\beta) = 0$, $\alpha, \beta \in [a, b]$, $\beta - \alpha \geq l$, then at least one point $\alpha + d < \xi < \beta - d$ satisfies $P'(\xi) = 0$. The function $d(l)$ is monotone increasing in l.*

Proof. We can assume that P is not identically zero on $[\alpha, \beta]$. Let ξ be the point on (α, β), where $|P(x)|$ attains its maximum $M = \|P\|_{[\alpha,\beta]}$. Then $P'(\xi) = 0$. On the other hand,

$$M = |P(\xi) - P(\alpha)| = (\xi - \alpha)\,|P'(\zeta)| \leq C_l \|P\|_{[\alpha,\beta]}(\xi - \alpha) = C_l M(\xi - \alpha).$$

Therefore $\xi - \alpha \geq C_l^{-1}$, and likewise $\beta - \xi \geq C_l^{-1}$. We select $d(l) = \min(C_l^{-1}, \frac{1}{2}l)$.

Remark. For algebraic polynomials of degree $\leqslant n$, the best value $d(l) = d_n(l)$ has been found by Turán [9]. If n is even, $d_n(l) = \frac{1}{2}l(1 - \cos \pi/n)$, and for any n, $d_n(l) \approx (l\pi^2/4)n^{-2}$.

For the system $S^{(k)}$, Theorem 8 and Lemma 9 produce a number $d_k(l)$. Taking $\delta(l) = \min_{0 \leqslant k < n} d_k(l)$, we obtain, for given S and n:

THEOREM 11. *There is a monotone increasing function $\delta(l)$, $0 \leqslant \delta(l) \leqslant \frac{1}{2}l$ such that if $\beta - \alpha \geqslant l$, $a \leqslant \alpha < \beta \leqslant b$ and $P^{(k)}(\alpha) = P^{(k)}(\beta) = 0$ for some P and k, $k = 0, \ldots, n-1$, then there exists a ξ, $\alpha + \delta(l) \leqslant \xi \leqslant \beta - \delta(l)$, for which $P^{(k)}(\xi) = 0$.*

5. INDEPENDENT SETS OF KNOTS

Let S be a system of functions on $[a, b]$, satisfying the assumption of Section 4. A set of knots $X = \{x_1, \ldots, x_m\} \subset [a, b]$ is called *independent with respect to* S, if for each interpolation matrix E, each polynomial P annihilated by E, X has a Rolle extension $\mathscr{R}$ with all new Rolle zeros ξ different from the x_k. As we know from Section 2, this $\mathscr{R}$ will be maximal and have no duplications. Lemma 3 yields then that the total number of (new and old) Rolle zeros of $P^{(k)}$ in $\mathscr{R}$ is exactly μ_k.

The construction of independent sets of knots is based on the technical lemma below. Without loss of generality, let $a = -1$, $b = 1$. We take $0 < y_1 < 1$ arbitrarily and choose y_j, $j = 2, 3, \ldots$ to increase rapidly to 1, with the following restrictions. If $\Delta(u) = \frac{1}{2}\delta(u)$, where $\delta(u)$ is the function of Theorem 11, then we require that $0 < y_{j-1} < y_j$ and

$$1 - y_j \leqslant \Delta^n(y_j - y_{j-1}), \qquad j = 2, 3, \ldots . \tag{5.1}$$

Since $\Delta(u) = \frac{1}{2}\delta(u) \leqslant 1$ and $\Delta(u)$ is increasing, it follows that

$$1 - y_j \leqslant \Delta(y_j - y_{j-1}) < \delta(y_j - y_{j-1}). \tag{5.2}$$

We also select numbers l_j satisfying

$$\Delta^{n-1}(y_j - y_{j-1}) \leqslant l_j \leqslant y_j - y_{j-1}, \qquad j = 2, 3, \ldots . \tag{5.3}$$

and put $l'_j = \Delta(l_j)$, $j = 2, 3, \ldots$; in general, l'_j are much smaller than l_j, for $\Delta(u) \leqslant \frac{1}{2}u$. We shall take X to contain some of the points $\pm y_i$, and will take care to select the new Rolle zeros ξ of X^k to be different from these points. This will follow because the ξ will be even outside of small intervals $(y_j - \varepsilon, y_j]$ or $[-y_j, -y_j + \varepsilon)$.

LEMMA 12. *Let $0 < \rho < 1$, and let s be so large that $\rho \leqslant y_s$, let $s + 2 \leqslant t$.*

Assume that P is a polynomial in S annihilated by E, X, and that the knots X and X^k for a Rolle extension $\mathscr{R}_k = (E^k, X^k)$ are contained in

$$[-\rho, \rho] \cup [y_t, 1] \cup [-1, -y_t] \tag{5.4}$$

but miss all intervals

$$(y_j - l_j, y_j), (-y_j, -y_j + l_j), \qquad j = t, t+1, \ldots .$$

Let $\rho < \rho' = y_{s-1}$. Then there is a Rolle extension (E^{k+1}, X^{k+1}) so that X^{k+1} is contained in the set

$$[-\rho', \rho'] \cup [y_t, 1] \cup [-1, -y_t] \tag{5.5}$$

and that the Rolle zeros selected between adjacent zeros of X^k miss the intervals

$$(y_j - l'_j, y_j], [-y_j, -y_j + l'_j), \qquad j = t, t+1, \ldots . \tag{5.6}$$

Proof. Because of the choice of ρ',

$$1 - \delta(\rho' - \rho) < \rho'. \tag{5.7}$$

Let $\alpha < \beta$ be two adjacent zeros of X^k. By means of Rolle's theorem, we shall find a zero ξ of $P^{(k+1)}$ of required kind.

(a) If $\alpha \leqslant \rho$, $\beta > \rho$, or $\alpha < -\rho$, $\beta \geqslant -\rho$, then ξ can be found in $(-\rho', \rho')$. Indeed, the length of (α, β) is at least $\rho' - \rho$. By Theorem 11, we find a Rolle zero ξ for which

$$\xi < \beta - \delta(\beta - \alpha) < 1 - \delta(\rho' - \rho) < \rho';$$

similarly, ξ satisfies $\xi > -\rho'$.

(b) The zeros of X^k fall into three groups: zeros cotained in $[-\rho, \rho]$, those in $[-1, -y_t]$ and those in $[y_t, 1]$. If α, β belong both to the first group, then (a) shows that we can take $\xi \in [-\rho', \rho']$. This is still true, again by (a), if α, β belong to different groups. If α, β belong both to the second or the third interval, then also ξ belongs to this interval.

(c) In the last case, we have still to show that ξ can be selected to miss (5.6). We can assume that $y_t \leqslant \alpha < \beta$. None of the intervals $(y_j - l_j, y_j)$ contains α or β, hence each of them is either contained in the interval (α, β) or disjoint with it. If the first possibility does not occur, we are through. In the opposite case, let j be the smallest integer $j > t$ for which $(y_j - l_j, y_j) \subset (\alpha, \beta)$. We shall find a $\xi < y_j - l'_j$, thus completing the proof. By Theorem 11 there is a ξ satisfying

$$\alpha < \xi \leqslant \beta - \delta(l_j) < y_j + (1 - y_j) - \delta(l_j).$$

From (5.1) and (5.3),

$$1 - y_i \leqslant \Delta(l_j) = \tfrac{1}{2}\delta(l_j).$$

Hence

$$\xi < y_j - \tfrac{1}{2}\delta(l_j) = y_j - l'_j.$$

COROLLARY 13. *Let the knots X be only among the points* $\pm y_j$, $j \geqslant s+2$, *or in* $[-\rho, \rho]$, *and suppose that the rows of E which correspond to knots* x_i, $-\rho \leqslant x_i \leqslant \rho$, *have no odd supported sequences. Then the construction of* X^{k+1} *of Lemma* 12 *will be without losses also in* $[-\rho, \rho]$, *with duplications possible only in this interval. Moreover, Rolle zeros in* $I_1 = [-1, -y_{s+2}]$ *or in* $I_2 = [y_{s+2}, 1]$ *will be produced by zeros only from the same interval; all other Rolle zeros will belong to* $I' = [-y_{s+1}, y_{s+1}]$.

We can apply Lemma 12, Coollary 13 and Lemma 2 to all derivatives $P^{(k)}$, $k = 0, \ldots, n-1$. In this way we obtain the following formulation of the method of independent knots.

THEOREM 14. *There exist numbers* $\rho = y_s$, ρ', $\rho < \rho' < 1$ *and an integer* $t > s$ *with the following properties. Let* $I = [-\rho, \rho]$, $I' = [-\rho, \rho']$, $I_1 = [-1, -y_t]$, $I_2 = [y_t, 1]$. *Let X be a subset of* $I \cup \{\pm y_t, \pm y_{t+1}, \ldots\}$, *and let E be an* $m \times (n+1)$ *interpolation matrix with no odd supported sequences in the rows corresponding to knots* $x_i \in I$. *Then each polynomial P in S annihilated by E, X has a maximal Rolle set* $\mathscr{R}$ *with duplication possible only in I. Moreover, all Rolle zeros are contained in* $I' \cup I_1 \cup I_2$; *those in* I_1 *(or in* I_2*) are produced only by zeros of the same interval; Rolle zeros produced with participation of one of the zeros in* I_1*(or* I_2*) lie in* $[-1, \rho]$ *(or in* $[-\rho, 1]$*). The total number of zeros of* $P^{(l)}$ *in* $\mathscr{R}$ *is equal to* μ_l *if X has no points in I. (If there is just one such point, Corollary* 4 *may apply).*

One can also assume that each P annihilated by E, Y, $Y \subset I' \cup \{\pm y_t, \ldots\}$ has a maximal Rolle set, if E has no odd supported sequences for knots in I'.

This is proved by applying Lemma 12 and Corollary 13 in turn to $P, P', \ldots, P^{(n-1)}$. At the kth step, we select $l_j = l_{jk} = \Delta^k(y_j - y_{j-1})$ and have $l'_j = \Delta(l_{jk}) = l_{j,k+1}$.

THEOREM 15. *For a given system S there exists an infinite sequence* $Y \subset [a, b]$ *with the property that each finite set* $X \subset Y$ *of m points is an independent set of knots for each* $m \times (n+1)$ *interpolation matrix.*

For Y we can take each of the two sets

$$\{\pm y_i\}_{i \geqslant r}, \qquad \{y_i\}_{i > r}.$$

The three points $-1 < x < 1$ are independent, if x is sufficiently close to -1 or 1.

6. Birkhoff Systems

It is interesting to investigate systems S for which Theorem B—the Atkinson–Sharma Theorem remains valid. We call $S = \{g_0, \ldots, g_n\}$ a *Birkhoff system if each Pólya matrix E which has no essential odd supported sequences, is regular with respect to S.*

Theorem 16. *A system $S = \{g_0, \ldots, g_n\}$ is a Birkhoff system if and only if equations*

$$P^{(k)}(x_k) = 0, \qquad a \leqslant x_k \leqslant b, k = 0, \ldots, n, \tag{6.1}$$

for a polynomial P in S imply $P \equiv 0$.

Proof. This is sufficient. Let P be annihilated by E, X, where E is a Pólya matrix, and let $E = E_1 \oplus \cdots \oplus E_\mu$ be its canonical decomposition into matrices without odd supported sequences. If the last column of E_λ is n_λ, we obtain from Lemma 3 that (6.1) is satisfied for $0 \leqslant k \leqslant n_1$. Next, $P^{(n_1+1)}$ is annihilated by E_2, and in the same way we get (6.1) for $n_1 < k \leqslant n_2$, and so on. Thus $P \equiv 0$.

The condition is necesaary, for Eqs. (6.1) mean that P is annihilated by a matrix (an Abel matrix), whose canonical decomposition consists of one column matrices.

Another form of condition (6.1) is that none of the determinants

$$V(x_0, \ldots, x_n) = \{g_0^{(k)}(x_k), \ldots, g_n^{(k)}(x_k), k = 0, \ldots, n\}, \qquad a \leqslant x_k \leqslant b, \tag{6.2}$$

should vanish. As a simple application of this, the system $S = \{1, \ldots, x^{k-1}, g_k, \ldots, g_n\}$ is a Birkhoff system exactly when $\{g_k^{(k)}, \ldots, g_n^{(k)}\}$ is a Birkhoff system.

There are relations between Chebyshev and Birkhoff systems.

Proposition 17. (i) *A Birkhoff system is an extended Chebyshev system*; (ii) *If for a system of functions S the Wronskian*

$$W(x) = \{g_0^{(k)}(x), \ldots, g_n^{(k)}(x), k = 0, \ldots, n\} \tag{6.3}$$

does not vanish identically, in particular if S is an extended Chebyshev system, then S is a Birkhoff system locally, that is, a Birkhoff system on some closed subinterval $[\alpha, \beta]$ *of* $[a, b]$.

Proof. (i) If a polynomial P in S has $n+1$ zeros, counting their multiplicities, then by Rolle's theorem one obtains (6.1), hence $P \equiv 0$, since S is a Birkhoff system. (ii) If $W(\bar{x}) \neq 0$ for some $\bar{x} \in [a, b]$, then the determinants (6.2) are different from zero if all x_k are close to $\bar{x}$.

EXAMPLE. $S = \{x^\alpha, x^\beta\}$, where $0 < \alpha \leqslant \beta$ is a Chebyshev system on $[a, b]$, $0 < a$ if and only if $\alpha < \beta$, and this is equivalent to regularity of 2×2 Birkhoff matrices with respect to S, but S is a Birkhoff system exactly when $(b/a)^{\beta - \alpha} < \beta/\alpha$.

A matrix E is *conditionally regular* for a system S on $[a, b]$ if one can find a set of knots $X \subset [a, b]$ which is regular with respect to E (for which the pair E, X is regular).

Remark 19. For Birkhoff matrices, we can complete the statements of Lemma 2 as follows: (a) If P is annihilated by a pair E, X which has a maximal Rolle extension, and if E satisfies the Pólya condition $M_l \geqslant l+1$, $0 \leqslant l \leqslant k_0$, then $P^{(l)}(z_l) = 0$ for some z_l, $0 \leqslant l \leqslant k_0$. If $k_0 = n$, then $P \equiv 0$. (b) The same happens if in addition to the assumption, also $P^{(k_0)}$ is annihilated by a pair Y, F with similar properties for $k_0 < l \leqslant n$. (c) For a Pólya matrix E, we have the regularity of the pair E, X in Theorem 14.

With Windhauer [10] we can apply independent knots to the study of conditional regularity.

THEOREM 20. *Each Pólya matrix E is conditionally regular with respect to a Birkhoff system S, and also with respect to a system S for which the Wronskian W(x) is not identically zero.*

Proof. Let S be a Birkhoff system on a subinterval $[\alpha, \beta]$ of $[a, b]$. By Theorem 15 we can find independent knots U: $u_1 < \cdots < u_m$ in $[\alpha, \beta]$. If P is annihilated by E, U, then this pair has a maximal Rolle extension, consequently $P^{(k)}(x_k) = 0$, $k = 0, \ldots, n$ for some $x_k \in [\alpha, \beta]$. Then $P \equiv 0$. Consequently, U is regular with respect to E. The second statement follows from Proposition 17(ii).

7. THE MAIN SINGULARITY THEOREMS

In this and the next section, we formulate and prove our main singularity theorems. We should mention that this proof can be somewhat simplified by coalesing the matrix E to three rows. In this way, intervals I_1, I_2 which appear below, would become single points $-1, +1$. The essential points of the proof would, however, remain unchanged.

A single one in a row of E (and this row itself) will be called a *singleton*. A singleton is an odd sequence, which may be supported in E or not. We begin with the simplest theorem:

THEOREM 21 (Lorentz and Zeller [7]). *A Birkhoff matrix is strongly singular if it contains a supported singleton.*

There is an immediate generalization:

THEOREM 22 (Lorentz [4]. *A Birkhoff matrix is strongly singular if it contains a row with precisely one odd supported sequence (all other sequences of this row being even or not supported).*

By the localization theorem (Proposition 17(ii)), both theorems hold also for extended Chebyshev systems. Thus, two localization theorems, Proposition 8 and 17 were used to obtain this conclusion.

We prefer to prove first Theorem 21, because this proof is simpler, illustrates our method. It is also (at least formally) not contained in the proof of the general Theorem 22. Throughout the proof the system $S = \{g_0, g_1, \ldots, g_n)$ will be a Birkhoff system on $[a, b]$.

Proof of Theorem 21. Let E be an $m \times (n+1)$ Birkhoff matrix which has a supported singleton $e_{i_0 q} = 1$ in the interior row i_0. We denote by E_0, E_1 matrices derived from E by omitting the row i_0, or by replacing it by $(1, 0, \ldots, 0)$ respectively. We use Theorem 14 and place knots x_i, $i < i_0$ into fixed independent positions in the interval $I_1 = [-1, -y_t]$, knots x_i, $i > i_0$ into similar positios in $I_2 = [y_t, 1]$. To this set of knots X_0 we add a variable knot $x \in I = [-\rho, \rho]$, to obtain the set X.

According to Theorem 14 and Remark 19 the pair E_1, X is regular, hence $P(x) = D(E_1, X) \neq 0$, $x \in I$. The function $P(x)$ is a polynomial in x and it is clearly annihilated by E_0, X_0. This pair has a maximal Rolle extension $\mathscr{R}$. We want to find a point $x = \xi \in I$ of $\mathscr{R}_q$ for which

$$P^{(q)}(\xi) = 0. \tag{7.1}$$

It is sufficient to find an $l < q$ for which $P^{(l)}$ has zeros of $\mathscr{R}$ both in I_1 and I_2, then the Rolle extension would produce a required ξ. If an l of this type would not exist, then for each $l < q$, Rolle zeros would be either all in I_1 or all in I_2. The Birkhoff condition, which is satisfied for $l < q$, yields at least two zeros of $P^{(l)}$. Let for $l = 0$ all of them be in I_1. Rolle's theorem produces a zero of P' in I_1, hence all Rolle zeros of P', and similarly for $P^{(l)}$, $l < q$ lie in this interval. This is impossible, since the supporting one from the right gives a zero in I_2.

Since $D(x) = D(E, X) = P^{(q)}(x)$, we see from (7.1) that E is singular. To establish strong singularity, we have to show that $D(x)$ changes sign at ξ. This is so because ξ is a simple zero of $P^{(q)} = D$. For otherwise we could add the one $e_{i_0,q+1} = 1$ to E, omitting a one in another now. By Remark 19, the new matrix and X would be a regular pair, and we would obtain $P \equiv 0$, a contradiction.

8. Proof of Theorem 22

After proving Theorem 21, we can assume that the row i_0 of the Birkhoff matrix E which contains a supported odd sequence, has at least two ones. Let $e_{i_0q} = 1$ be the first one of the sequence. We denote by E_0 the matrix obtained from E by replacing this one by zero, by $\bar{E}'$, E', $\bar{E}''$, E'' matrices consisting of rows $i < i_0$, $i \leqslant i_0$, $i > i_0$, $i \geqslant i_0$ of E_0, and by $\bar{\mu}'_l$, μ'_l, $\bar{\mu}''_l$, μ''_l, μ_l the functions μ of Section 2 for the last five matrices. Obviously, $\bar{\mu}'_l \leqslant \mu'_l$, $\bar{\mu}''_l \leqslant \mu''_l$.

To be able to use Theorem 14, we assign to x_i, $i < i_0$ and $i > i_0$ independent positions $\pm y_j$ in the intervals I_1, I_2. To this set X_0 we add the knot $x_{i_0} = x$ in $I = [-\rho, \rho]$; let $X = X_0 \cup (x)$. In addition, let $-\rho' \leqslant y \leqslant \rho'$, for $y \notin X$ let $Y = X \cup (y)$, further let E_1 be the matrix obtained from E_0 by adding the row $(1, 0, ..., 0)$ between rows i_0 and $i_0 + 1$. Then $P(x, y) = D(E_1, Y)$ is a polynomial in x and y. For a fixed x, it is a polynomial $P(y)$ in y, and the structure of the determinant shows that $P(y)$ is annihilated by E_0, X. Since the matrix E_1 has no odd supported sequences for knots in $[-\rho', \rho']$, the pair E_1, Y is regular by Remark 19(c); thus $P(y) = 0$ for $y \neq x$.

We consider the derivative $\partial^q P/\partial^q y = P^{(q)}(y) = P^{(q)}(x, y)$. Since

$$P^{(q)}(x, x) = D(E, X), \tag{8.1}$$

singularity of E will be established, if we show that for some x, the equation

$$P^{(q)}(x, y) = 0 \tag{8.2}$$

is satisfied for $y = x$. We are thus led to consider solutions y of (8.2) for fixed x. We can say at once that for $x = \rho$ (or $x = -\rho$) this equation has no solution $y = \rho$ (or, correspondingly, $y = -\rho$). For if $x = \rho$, X is independent, and $D(E, X) = P^{(q)}(\rho, \rho) \neq 0$ by Remark 19(c).

By Theorem 14, there is a maximal Rolle extension $\mathscr{R}$ of the pair (E_0, X) that annihilates P. For this extension, Rolle zeros y produced by pairs of knots other than those confined to I_1 or to I_2, lie in $[-\rho', \rho']$. This explains our choice of the domains of the variables, $-\rho \leqslant x \leqslant \rho$, $-\rho' \leqslant y \leqslant \rho'$.

Let t be the number of solutions of (8.2) for a given x, in other words, the number of Rolle zeros of $P^{(q)}$ in $[-\rho', \rho']$. Rolle zeros of $P^{(l)}$ of $\mathscr{R}$ in the intervals I_1, I_2, are produced (by Theorem 14) by knots in these intervals and by matrices $\bar{E}'$, $\bar{E}''$. Their numbers are $\bar{\mu}'_l$, $\bar{\mu}''_l$, $l = 0,..., n$. One cannot claim that μ_l is the total number of Rolle zeros of $P^{(l)}$. However, for $l = q$ this is true by Corollary 4. Hence

$$t = \mu_q - \bar{\mu}'_q - \bar{\mu}''_q. \tag{8.3}$$

We see that t is independent of x. We also have $t \geqslant 1$. This can be deduced by the argument in the proof of Theorem 21. Alternatively, we have

$$t \geqslant \sigma = \mu_q - \mu'_q - \mu''_q, \tag{8.4}$$

and by Theorem 5, $\sigma \geqslant 1$.

Let

$$-\rho' \leqslant y_1(x) < \cdots < y_t(x) \leqslant \rho', \qquad -\rho \leqslant x \leqslant \rho, \tag{8.5}$$

be all Rolle zeros of $P^{(q)}(y)$ contained in $[-\rho', \rho']$. We claim: (a) $P^{(q)}(y)$ has no other zeros in $[-\rho, \rho]$; (b) each of the zeros (8.5) is simple except perhaps the zero $y_0(x) = x$; (c) If there is a zero $y_0(x) = x$, it has an odd multiplicity (equal to the length of the sequence containing $e_{i_0q} = 1$). Indeed, E_0 is an $m \times n$ Pólya matrix, and in the Rolle extension (E_0^q, X^q), E_0^q is a Pólya matrix with $n - q$ columns (and zero column numbered $n + 1 - q$). If one of the above statements were not true, we would be able to add to E_0^q an additional one, obtaining a new Pólya matrix with $n + 1 - q$ ones and columns, which annihilates $P^{(q)}$ and has no odd supported sequences for knots in $[-\rho', \rho']$. By Remark 19(b) we would obtain $P \equiv 0$, a contradiction.

Since the function $P(x, y)$ is continuous, it is now easy to prove the continuity of the zeros (8.5).

To prove the singularity of E, we have to show that for some s, one has $y_s(x) = x$, for a certain $x \in I$, in other words, that one of the curves (8.5) in the rectangle $-\rho \leqslant x \leqslant \rho$, $-\rho' \leqslant y \leqslant \rho'$ intersects the line $y = x$ (see Fig. 1). It is not obvious that this intersection exists, for there are intervals on the lines $x = \pm\rho$ through which the curves could escape. [This remark applies also to the proof which uses coalescence to three rows. If x, y charge in the open interval $(-1, 1)$, the intersection must lie in the open square. The curves still could escape through the corners of the square—a point missed in [2]].

Let $N(x)$, $-\rho \leqslant x \leqslant \rho$ be the number of y_s which satisfy $y_s > x$. We can find $N(\rho)$: this is the number of Rolle zeros ξ of $P^{(q)}(\rho, y)$ which satisfy $\rho < \xi \leqslant \rho'$, or equivalently $\rho \leqslant \xi \leqslant \rho'$. Now Rolle zeros ξ in $[\rho, 1]$ are produced by the matrix E'' and the independent knots $\rho = x_{i_0},..., x_m$. Their number (by Theorem 14) is μ''_q. The ξ with $\xi > \rho'$ are produced by $\bar{E}''$ and the knots $x_{i_0+1},..., x_m$, there are $\bar{\mu}''_q$ of them. Hence

$$N(\rho) = \mu''_q - \bar{\mu}''_q.$$

FIGURE 1

Next let $x = x_{i_0} = -\rho$. The number of $y_s > -\rho$ is equal to the number of Rolle zeros ξ of $P^{(q)}(-\rho, y)$ with $-\rho \leqslant \xi \leqslant \rho'$. Here again, the knots $x_1, \ldots, x_m$ are in independent positions, and we can find the numbers of different types of ξ by means of Theorem 14. Their total number is μ_q. Zeros $\xi < -\rho$ are produced exclusively by the knots $x_1, \ldots, x_{i_0} = -\rho$ and the matrix E', their number is μ_q'. The number of ξ with $\xi > \rho'$ is $\bar{\mu}_q''$. It follows that

$$N(-\rho) = \mu_q - \mu_q' - \bar{\mu}_q''.$$

This yields

$$N(-\rho) - N(\rho) = \mu_q - \mu_q' - \mu_q'' = \sigma. \tag{8.6}$$

By Theorem 5, $\sigma \geqslant 1$. Thus, at lest σ curves $y_s(x)$ cross the line $y = x$ inside te interval $[-\rho, \rho]$. The curves (8.5) divide the rectangle $-\rho \leqslant x \leqslant \rho$, $-\rho' \leqslant y \leqslant \rho'$ into $t + 1$ regions with $P^{(q)}(x, y)$ alternating in sign from region to region. The point (x, x) moving on the line $y = x$ crosses $\sigma + 1$ of the regions; the points $(-\rho, -\rho)$, (ρ, ρ) are not on the curves. This means that $D(E, X)$ changes sign σ times as x moves on $[-\rho, \rho]$.

9. Historical Notes

1. I have communicated a proof of Theorem 22 (for ordinary singularity) to K. Zeller in the Summer of 1969 and have presented it at the Annual Meeting of the AMS in January 1970 in New Orleans. An abstract has appeared in the November 1969 issue of the *Notices of the American Mathematical Society* [4]. My paper [3], with the proof, has been submitted to the *Journal of Approximation Theory* in March 1970 (written communication of O. Shisha to me); this date does not appear on the paper

itself. Unfortunately, paper [2] of Karlin and Karon, submitted in December 1970, appeared in the *Journal of Approximation Theory* earlier than [3].

2. In April 1970 I was invited to Stanford to give some lectures. There Professor S. Karlin told me that he was about to prove the "Atkinson–Sharma conjecture" (that a Birkhoff matrix with an odd supported sequence is singular). I showed him the preprints of [7] and [3]. Later he told me that he could prove Theorem 22 more simply by his new method (which he did not describe to me). I have received from him a proof in writing in December 1970. The proof presented in this paper was written down in 1974 and appeared in 1975 [5].

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